Remark on Average Velocity

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The following puzzle has been around for many years in various forms: Say that a motorist completes the first half of a trip at an average velocity of 20 miles per hour(mph.) Is it possible, even in principle, to cover the second half at a velocity such that the average for the entire journey is 40 mph?

It is easy to see that the answer is "no." Suppose, for definiteness, that the entire trip is 40 miles. The driver must then have consumed an entire hour in the first half, making it impossible to complete the journey within the required one hour's time. (The distance of the trip is irrelevant; the same conclusion holds no matter what the distance. Moreover, all that is important about the two average velocity figures is that their ratio be equal to the fraction of the trip covered at the lower velocity.)

What is interesting about the puzzle is that people tend to jump to the wrong conclusion, perhaps because they confuse the average over time of the velocity with its average over space. More precisely, let $v(t)$ be the velocity during time interval $[0, T]$ of an object moving in one dimension. Assuming $v(t)$ is smooth enough for the integral to be defined, the usual interpretation of *average velocity* is the quantity $\bar{v}$ defined by

$$\bar{v} \;=\; \frac{1}{T}\int_0^T v(t)\,dt.$$

If in addition $v(t) > 0$, then $x(t) = \int_0^t v(s)ds$ is a strictly increasing function. It is then possible to invert the relationship by expressing t as a function of x: $t = t(x), 0 \leq t \leq L$, where L is the total distance travelled.

In the example above, if it were possible to travel the second half at infinite velocity, then the average velocity would be exactly 40 mph. One could argue on this basis that the average over x is more natural. After all, the usual time average obscures something rather important about the velocity – the fact that it becomes infinite! Accordingly, let us introduce the spatial average, $\hat{v}$, as

$$\hat{v} \;=\; \frac{1}{L}\int_0^L v(x)\,dx.$$

A numerical example may help to clarify the definitions.

Example: Let $v(t) = t^2, 0 \leq t \leq 1$. (Here $T = 1$.) Then $x(t) = \frac{1}{3}t^3$, and $L = \frac{1}{3}$. Therefore $\bar{x} = \frac{1}{3}$, while $\hat{x} = \frac{3}{5}$. (Here $t(x) = (3x)^{\frac{1}{3}}$.)

Which of $\frac{3}{5}$ or $\frac{1}{3}$ is the more meaningful figure for average velocity in this example is debatable, but at least the two notions of average velocity interact in a way that is aesthetically pleasing:

Theorem: Assume the velocity is continuous and positive. Then

$$\bar{v}\hat{v} \quad = \quad \frac{1}{T}\int_0^T v^2(t)\,dt.$$

To put it another way, the (normalized) L^2 norm of velocity is the geometric mean of $\bar{v}$ and $\hat{v}$.

Proof: Since $v = \frac{dx}{dt}$, the substitution $x = x(t)$ yields

$$\int_0^T v^2(t)\,dt = \int_0^T v(x(t))\frac{dx}{dt}\,dt = \int_0^L v(x)\,dx.$$

The result follows after dividing by $T = \frac{L}{\bar{v}}$.

Physically, $\hat{v}$ is double the ratio of (time) average kinetic energy to (time) average linear momentum.

We have not attempted to state the most general possible result. No doubt the smoothness and positivity assumptions on v can be relaxed considerably.