Slant Asymptotes and the Legendre Transform

Terry R. McConnell
Syracuse University

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Abstract

We derive a formula for the y-intercept of an asymptote to the graph of a function.

1 Main Result

A (non-vertical) *asymptote* of the graph of a function $f(x)$ is an affine linear function $y = mx + b$ such that the difference $f(x) - (mx + b)$ tends to zero as x tends to either plus or minus infinity. When the slope m equals zero it is customary to call such asymptotes *horizontal asymptotes*, and when $m \neq 0$ they are termed *slant asymptotes.*

Calculus students identify horizontal asymptotes by finding the limit of $f(x)$ (if it exists) as x tends to plus or minus infinity. Slant asymptotes are somewhat more exotic, and most modern texts are content to observe that the graphs of rational functions exhibit such asymptotes when the degree of the numerator exceeds by one the degree of the denominator. The affine asymptote can then be found by polynomial long division, a skill that is seldom taught anymore. Older texts sometimes gave more detailed treatments. See, for example, [1], pp 250-253.

In the case of non-rational functions, one suspects the presence of a slant asymptote when the derivative has a non-zero horizontal asymptote, although this condition is not generally sufficient (see below.) When a function does have a slant asymptote, its slope can readily be found as the limiting value of the derivative, but finding the y-intercept is less straightforward. In the following result we provide a formula for the y-intercept that is sufficiently general to handle most cases of interest. (In the concluding section of the paper we discuss generalizations and examples.)

Theorem 1.1. *Let f be twice continuously differentiable. Then the following two conditions are equivalent:*

$$\lim_{x\to\infty} f(x) - xf'(x) = b \text{ exists.} \tag{1.1}$$

$$\int_0^\infty xf''(x)\,dx \quad \text{is convergent.} \tag{1.2}$$

If, moreover,

$$\int_0^\infty x|f''(x)|\,dx < \infty, \tag{1.3}$$

then the limit $m = \lim_{x\to\infty} f'(x)$ exists, and the graph of $y = f(x)$ has asymptote $y = mx + b$ as $x \to \infty$.

For the proof, we have that

$$\int_y^z xf''(x)\,dx = f(y) - yf'(y) - (f(z) - zf'(z)).$$

Thus, (1.1) and (1.2) are equivalent by the Cauchy criterion. Next, note that for any $0 \le A \le B < \infty$,

$$|f'(B) - f'(A)| \le \frac{1}{A} \int_A^B x|f''(x)|\,dx.$$

If (1.3) holds, then existence of the finite limit $m = \lim_{x\to\infty} f'(x)$ again follows from the Cauchy criterion.

Now $f(x) - mx = f(x) - xf'(x) - x(m - f'(x))$. But for $x \ge 0$,

$$0 \le x|m - f'(x)| = x\left|\int_x^\infty f''(u)\,du\right| \le \int_x^\infty u|f''(u)|\,du.$$

Since the latter tends to zero as $x \to \infty$ by (1.3), we conclude that $b = \lim_{x\to\infty} f(x) - mx$, and the proof is complete.

To explain the title of the paper, recall that the *Legendre transform* of a strictly convex function f is the function h defined by

$$h(y) = f(x) - xf'(x), \quad y = f'(x).$$

More directly, if g denotes the inverse function of the derivative of f, then $h(y) = f(g(y)) - yg(y)$. The Legendre transform gives the y-intercept of a line tangent to the graph of f as a function of its slope. It is an important tool in classical mechanics, thermodynamics, and more recently, the probabilistic theory of large deviations. For convex functions an asymptote may be viewed as a tangent line at infinity, and its intercept is the limiting value of the Legendre transform.

If we assume that a twice continuously differentiable convex function has an asymptote $mx + b$ at $+\infty$ then $f(x) - xf'(x) \ge f(x) - mx \ge b$, and condition (1.1) holds since the function $f(x) - xf'(x)$ is non-increasing. Therefore the conclusions of the theorem hold, and in particular $m = \lim_{x\to\infty} f'(x)$. On the other hand, this latter condition is not sufficient to have an affine linear asymptote, as can be seen from the example $f(x) = x - \ln(x + 1)$.

2 Complements

If the function f in Theorem 1.1 is known to be either convex or concave for all sufficiently large x then (1.3) follows from (1.2) or (1.1). In particular, asymptotes to algebraic curves can always be found by attempting to compute the limit (1.1). Here is a sketch of the required argument: A plane algebraic curve is the solution set of a polynomial equation $P(x, y) = 0$. The function or functions $y = y(x)$ obtained by solving this equation for y can have at most finitely many zeros since these values of x are roots of the polynomial $P(x, 0)$. Implicit differentiation leads to a formula $y' = R(x, y)$, where R is a certain rational function. If we denote by Q the polynomial in the numerator, then values of x where $y'(x) = 0$ correspond to crossing points of the two algebraic curves $P(x, y) = 0$ and $Q(x, y) = 0$, and there are at most finitely many such points by Bezout's Theorem. Repeating the argument for the second derivative shows that it too can have at most finitely many roots, and therefore has constant sign for sufficiently large x. (I thank Fred Richman for pointing that it is unnecessary to appeal to Bezout's Theorem. Implicit differentiation shows that y' is algebraic over the field $\mathbb{R}(x, y)$ of real valued rational functions in the variables x and y. But y is algebraic over $\mathbb{R}(x)$, hence y' is also algebraic. A similar argument shows that y'' is algebraic, and so has at most finitely many zeros.)

As an example, consider the curve $y^3 = 2ax^2 - x^3$ analyzed on p. 253 of [1]. We have

$$f'(x) = \frac{4ax - 3x^2}{3(2ax^2 - x^3)^{2/3}},$$

and this expression has limit $m = -1$ as $x \to \infty$. We also have that

$$f(x) - xf'(x) = \frac{2ax^2}{3(2ax^2 - x^3)^{2/3}},$$

and this expression has limit $b = 2a/3$ as $x \to \infty$. Thus the curve has slant asymptote $y = -x + \frac{2}{3}a$ at plus infinity.

As a second example, consider the *Folium of Descartes* with equation $y^3 + x^3 - 3axy = 0, a > 0$. The curve is unbounded since for any given x the cubic in y has at least one real root. An equivalent form of the equation is

$$x + y = \frac{3a}{\frac{y}{x} - 1 + \frac{x}{y}}.$$

Since $|u + 1/u| \geq 2$ for any real number u, we must have $|x+y| \leq 3a$, and hence $y/x \to -1$ as either $|x| \to \infty$ or $|y| \to \infty$. Implicit differentiation gives the formula

$$y - xy' = \frac{axy}{y^2 - ax} = \frac{a\frac{y}{x}}{(\frac{y}{x})^2 - \frac{a}{x}}.$$

Thus, we conclude that the folium has asymptote $y = -x - a$, a result first obtained by Huygens.

Under the conditions of Theorem 1.1 we have the following integral representation of the y-intercept of an asymptote at infinity:

$$b = f(A) - mA - \int_A^\infty (x - A) f''(x)\, dx. \tag{2.1}$$

This follows from (1.1) and integration by parts. Finally, we note that the y-intercept has a pleasant interpretation as the first moment of a measure in the general convex case.

Theorem 2.1. *Let f be a convex function on $(0, \infty)$ such that $f(0) = 0$. Then the second derivative of f in the sense of distributions is given by a positive Radon measure μ. If*

$$\int_0^\infty x\, \mu(dx) < \infty$$

then the graph $y = f(x)$ has an asymptote $y = mx + b$ as $x \to \infty$, where

$$m = \lim_{x \to \infty} f'(x), \quad b = \lim_{x \to \infty} f(x) - xf'(x) \quad = \quad -\int_0^\infty x\, \mu(dx).$$

Here we take for f' any non-decreasing function equal almost everywhere to the derivative of f (which exists in the classical sense, almost everywhere.) We may take for μ the Lebesgue-Stieltjes measure for which

$$f'(z) - f'(y) = \int_y^z \mu(dx) \tag{2.2}$$

holds for almost every choice of y and almost every choice of z. The identity

$$\int_y^z x\, \mu(dx) = f(y) - yf'(y) - (f(z) - zf'(z))$$

then holds for almost every choice of y and almost every choice of z. (Write x as an integral and use (2.2),Fubini's Theorem, and the fact that a convex function is absolutely continuous.) The proof of Theorem 1.1 can then be repeated with a few obvious modifications.

References

[1] W.A. Granville , *Elements of the Differential and Integral Calculus*, Ginn and Company, Boston, 1911.