Note on Segmented Series

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Abstract

We consider some examples of conditionally convergent series.

1 Introduction

Conditionally convergent series are infinite series that converge but do not converge absolutely. Apart from the Alternating Series Theorem, there are surprisingly few general methods for constructing examples of conditionally convergent series. In the next section we present a method for constructing examples based upon a slight extension of the results of [6].

2 Conditionally Convergent Series Examples

Following [6] we shall say that a series

$$\sum_{j=1}^{\infty} a_j$$

is *segmentally alternating* if there is a strictly increasing sequence j_n with $j_1 = 1$ such that for each n, half the terms of the *segment* $a_{j_n}, a_{j_n+1}, \ldots a_{j_{n+1}-1}$ are positive and half are negative. (This requires that segments have even length.) If in addition we have $|a_n| \searrow 0$ then the series is said to be segmentally alternating of Leibniz type (SALT). Among other things, the authors of [6] prove that if the segments of a SALT series have bounded length (i.e, there is k such that $j_{n+1} - j_n \leq k$ for all n) then the series converges, possibly conditionally.

We remark first that the Leibniz assumption $|a_n| \searrow 0$ can be relaxed to $M_{n+1} \leq m_n$, with M_n tending to zero, where M_n denotes the maximum $|a_j|$ in the nth segment, and m_n denotes the minimum. If this is so, then the terms of each segment can be rearranged to form the terms of a SALT series. The resulting rearranged series converges by Theorem 1 of [6], and hence the original series converges by Theorem 8.14 of [1]. (I thank Robert Kantrowitz for pointing out references [1], [3], and [4], and for several interesting discussions.)

A number μ will be called a *median* of a segment of terms if at least half the terms a_j in the segment satisfy $a_j \leq \mu$ and at least half satisfy $a_j \geq \mu$. If the segment has odd length then μ is unique and equal to the middle term when terms are arranged in increasing order. If the segment has even length then any number between or equal to one of the two middle terms can be taken as a median.

Theorem 2.1. *Let μ_n be a median of the nth segment of a series $\sum_{j=1}^{\infty} a_j$. Assume that the series*

$$\sum_{n=1}^{\infty} \mu_n$$

converges if the segments have constant length, and converges absolutely if the segments have non-constant but bounded length. Also assume that

$$\max\{|a_j - \mu_{n+1}| : j_n \le j < j_{n+1}\} \le \min\{|a_j - \mu_n| : j_{n-1} \le j < j_n\}, n = 2, 3, \dots$$

and that these maxima and minima have limit zero as n tends to infinity. Then $\sum_{j=1}^{\infty} a_j$ converges, possibly conditionally.

Proof. Let

$$b_j = a_j - \mu_n,\ j_n \le j < j_{n+1}.$$

Then, as noted above, the terms b_j in each segment can be rearranged to give the terms of a SALT series. This series converges by Theorem 1 of [6], and then $\sum_{j=1}^{\infty} b_j$ converges by Theorem 8.14 of [1]. Let k_n be the length of the nth segment. Then $\sum_{n=1}^{\infty} k_n\mu_n$ converges by the comparison test, since the k_n are bounded, and the desired result follows.

Under the hypotheses of Theorem 1 of [6] one may take all medians equal to zero.

Using Theorem 2.1 we can construct many examples of conditionally convergent series. For example, for the nth segment introduce numbers λ_j having 0 as a median and such that

$$\max\{|\lambda_j| : j_n \le j < j_{n+1}\} \le \min\{|\lambda_j| : j_{n-1} \le j < j_n\}.$$

These maxima and minima must also tend to zero as n tends to infinity. Define $a_j = \mu_n + \lambda_j$ for j in the nth segment. Then if segment lengths are bounded and the series of μ_n is absolutely convergent, or if segment lengths are constant and the series of μ_n is convergent, the series of the a_j converges, possibly conditionally.

In the constant segment length case there is the possibility of proceeding recursively by representing the series of the μ_n as another such series with its own segment length, and so forth.

Theorem 2.2. *Let*

$$\sum_{j=1}^{\infty} \lambda_j^{(k)}$$

be a segmentally alternating series for each $k = 0, 1, \dots$, with nth segment $r_n^{(k)} \le j < r_{n+1}^{(k)}$ such that $r_{n+1}^{(k)} - r_n^{(k)} \le R_k < \infty$. Assume that

$$|\lambda_j^{(k)}| \le L_k < \infty, \tag{2.1}$$

for each j. Also assume that for each k the $\lambda_j^{(k)}$ tend to zero as $j \to \infty$ and satisfy inequalities

$$\max\{|\lambda_j^{(k)}| : r_n^{(k)} \le j < r_{n+1}^{(k)}\} \le \min\{|\lambda_j^{(k)}| : r_{n-1}^{(k)} \le j < r_n^{(k)}\},$$

and that

$$\sum_{k=0}^{\infty} R_k L_k < \infty. \tag{2.2}$$

Define a sequence a_j by

$$a_j = \sum_{k=0}^{\infty} \lambda_j^{(k)}.$$

Then $\sum_{j=1}^{\infty} a_j$ converges, possibly conditionally.

For the proof, note that, by Theorem 2.1, $\sigma_k = \sum_{j=1}^{\infty} \lambda_j^{(k)}$ converges for each fixed k and satisfies

$$|\sigma_k| \le \frac{1}{2} R_k L_k. \tag{2.3}$$

The same upper bound holds for all the partial sums indexed by one of the $r_n^{(k)}$. This essentially follows from Theorem 2 of [6], since terms can be rearranged within segments to form the sequence of terms of a SALT series without affecting the partial sums in question. (Theorem 2 of [6] requires segments of constant length, but that assumption is unnecessary for the present conclusion: one can simply pad each segment as needed to the maximum length by appending terms of alternating sign and equal in absolute value to the smallest absolute value term in the segment. Again this does not affect the relevant partial sums.)

Lemma 2.1. *Suppose the series*

$$\sum_{j=1}^{\infty} \lambda_j^{(k)} = \sigma_k$$

converge for every k. Also suppose we have bounds on the maximal partial sums

$$\sup_n \left| \sum_{j=1}^{n} \lambda_j^{(k)} \right| \leq M_k,$$

with

$$\sum_{k=0}^{\infty} M_k < \infty.$$

Let $a_j = \sum_{k=0}^{\infty} \lambda_j^{(k)}$ and $s = \sum_{k=0}^{\infty} \sigma_k$. Then the series

$$\sum_{j=1}^{\infty} a_j$$

converges to s.

For the proof, given $\epsilon > 0$, first choose N so large that

$$\sum_{k=N+1}^{\infty} M_k \ < \ \frac{\epsilon}{2}.$$

For any M,

$$|s - \sum_{j=1}^{M} a_j| \leq \sum_{k=0}^{\infty} |\sigma_k - \sum_{j=1}^{M} \lambda_j^{(k)}| < \epsilon + \sum_{k=0}^{N} |\sigma_k - \sum_{j=1}^{M} \lambda_j^{(k)}|,$$

and we finish by taking M sufficiently large.

Returning to the proof of Theorem 2.2, note that we have the bound

$$\sup_n \left| \sum_{j=1}^{n} \lambda_j^{(k)} \right| \leq R_k L_k. \tag{2.4}$$

To see this, recall that $\frac{1}{2} R_k L_k$ is an upper bound for partial sums indexed by $n = r_i^{(k)}, i = 1, 2, \ldots$ by the remarks following (2.3). For values of n in between, the worst case occurs when all terms of the same sign occur at the beginning of a segment. Theorem 2.2 then follows from (2.2) and Lemma 2.1.

Example 2.1 ([3], Problem 3.5.9) The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

converges to ln(2). The following rearrangement of the alternating harmonic series converges to exactly half this value:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$

Convergence of the series can be seen from Theorem 2.1 after replacing each positive term by two equal terms and using segments of length 4. To evaluate the sum, group the terms of the original series into segments of length 3 and note that the general term of the resulting series is given by

$$\frac{1}{2n-1} - \left(\frac{1}{4n-2} + \frac{1}{4n}\right) = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n}\right).$$

The following rearrangement of the alternating harmonic series, on the other hand, converges to the same sum as the alternating harmonic series itself:

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \dots$$

Convergence follows from Theorem 2.1, using segments of length 6. Since partial sums indexed by multiples of six agree with the corresponding partial sums of the alternating harmonic series, both series have the same sum.

Example 2.2. The pattern of alternation of sign in the terms of the series of Example 2.1 is periodic. Here we construct non-periodic examples based upon the *Thue-Morse sequence*. The Thue-Morse sequence may be obtained by starting with the finite sequence 01 and then repeatedly applying the substitution that replaces each 0 with 01, and each 1 with 10. When iterated indefinitely, this substitution produces an infinite sequence of binary digits that begins 0110100110010110 . . . . (A convenient reference for properties of the Thue-Morse sequence is [5].)

The binary digits in the construction of the Thue-Morse sequence can be replaced by any other pair of distinct symbols. If we let 1 play the role of 0 and -1 play the role of 1, then the resulting sequence t_n alternates in sign in a way that is not periodic, but is nevertheless quite balanced with regard to the numbers of positive versus negative terms. A more direct formula for the nth term of the resulting sequence can be given in terms of the binary expansion of n: Let $b(n)$ denote the number of 1 digits in the binary expansion of n. Then $t_n = (-1)^{b(n)}$. See, e.g., Proposition 2.2.2 of [5].

Theorem 2.3. *Let $d_n, n = 0, 1, 2, \dots,$ be a non-decreasing unbounded sequence of positive real numbers. Then the series*

$$\sum_{n=0}^{\infty} (-1)^{b(n)} \frac{1}{d_n}$$

converges, possibly conditionally.

For the proof, one only needs to note that the substitution used in the construction of the Thue-Morse sequence, when applied to segments of symbols 1 and -1, preserves the property of having equal numbers of positive and negative terms in a segment. Then apply Theorem 1 of [6] with segment length equal to any power of $2 : 2, 4, 8, \dots$

If we use Theorem 2.1 above in place of Theorem 1 of [6], then we may freely permute the $\frac{1}{d_n}$ in blocks of length equal to a fixed power of 2, and the resulting series still converge.

It would be of interest to find the exact value of

$$\sum_{n=0}^{\infty} (-1)^{b(n)} \frac{1}{n+1} = 0.398761 \dots$$

Example 2.3. This extended example is motivated by the Fourier series,

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n \alpha)}{n},$$

which converges for all choices of the real number α. The sequence $\frac{1}{n}$ here can be replaced by any decreasing sequence with limit zero. Convergence follows from partial summation and the following rather remarkable property of the sine function: for any fixed real number α there is a finite constant M depending only on α such that

$$\left|\sum_{j=1}^{n} \sin(2\pi j\alpha)\right| \leq M, n = 1, 2, \ldots \tag{2.5}$$

If α is rational, then the sequence $\sin(2\pi j\alpha)$ is periodic and the sum over a full period is zero. If α is irrational, then $|1 - e^{2\pi i\alpha}| > 0$, and (2.5) follows by noting that the series there is the imaginary part of the sum of the geometric series with terms $e^{2\pi ij\alpha}$.

Motivated by this result, we may ask whether other functions than the sine function have a similar property? Let f be a periodic function of period 1. (The choice of period is immaterial, but period 1 is convenient for the present discussion, since the choice $f(x) = \sin(2\pi x)$ has that period.) For which such functions f is the sequence

$$\sum_{j=1}^{n} f(j\alpha), n = 1, 2, \ldots$$

a bounded sequence for each choice of real number α?

If we assume that f is Lebesgue integrable, then a necessary condition is that f should have mean zero, i.e.,

$$\int_0^1 f(x)\, dx = 0.$$

This is an immediate consequence of the following well-known and celebrated result:

Theorem 2.4. *(Kronecker-Weyl-Birkhoff) If f has period 1 and is Lebesgue integrable over finite intervals, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(j\alpha) = \mu,$$

for almost every α, where

$$\mu = \int_0^1 f(x)\, dx.$$

If f is assumed to be Riemann integrable, then we have convergence for *every* real number α, and convergence to μ for every irrational α. (See, e.g., [2], especially example 7.8 and appendix 9.) Accordingly, we shall say that a Riemann integrable function f having mean zero and period 1 is **KWB** if (2.5) holds with $f(j\alpha)$ in place of $\sin(2\pi j\alpha)$.

It follows from (2.5) that any function of the form

$$f(x) = \sum_{j=1}^{n} A_j \sin(b_j x),$$

for real numbers $A_j, b_j, j = 1, 2, \ldots, n$ is KWB.

Consider in place of the sine function the function $\phi(x)$ of period 1 defined as $+1$ for $0 < x < \frac{1}{2}$, -1 for $\frac{1}{2} < x < 1$, and equal to zero whenever $2x$ is an integer. As an application of Theorem 2.2 we prove the following:

Theorem 2.5. *Let c_k be a sequence of real numbers satisfying*

$$\sum_{k=1}^{\infty} k|c_k| < \infty,$$

and define

$$f(x) = \sum_{k=1}^{\infty} c_k \phi\left(\frac{x}{k}\right).$$

Then for any $\alpha \in \mathbb{Q}$ *we have*

$$\sup_n \left| \sum_{j=1}^{n} f(j\alpha) \right| < \infty.$$

By Kronecker's Lemma, it suffices to show that the series

$$\sum_{j=1}^{\infty} \frac{f(j\alpha)}{d_j}$$

converges whenever $0 < d_j \nearrow \infty$. Thus, fix such a sequence d_j and $\alpha \in \mathbb{Q}$. Let

$$\lambda_j^{(k)} = c_k \frac{\phi(j\alpha/k)}{d_j}, k, j \in \mathbb{N}.$$

Since α/k is rational, say $\alpha/k = \frac{p}{kq}, p, q \in \mathbb{Z}$, the series

$$\sum_{j=1}^{\infty} \lambda_j^{(k)}$$

is segmentally alternating with segment length bounded by $k|q|$. Also, we have $|\lambda_j^{(k)}| \leq \frac{|c_k|}{d_1}$. Since

$$\frac{f(j\alpha)}{d_j} = \sum_{k=1} \lambda_j^{(k)},$$

the desired result follows from Theorem 2.2.

We have not been able to determine whether the function ϕ is KWB, or indeed if there are any examples of KWB functions besides trigonometric polynomials.

Example 2.4. As another application of Theorem 2.2, we give a proof of the main result of [4].

Theorem 2.6. *Let* $b_1 \geq b_2 \geq \dots$ *be a sequence of real numbers with finite real limit* c*. Also assume that for each* $n \geq 0$ *there are real numbers* $a_{n,j}, j = 1, 2, \dots L$ *such that* $|a_{n,j}| \leq M < \infty$ *for all* n *and* j*, and*

$$\sum_{j=1}^{L} a_{n,j} = 0.$$

Then the series

$$\sum_{n=0}^{\infty} \left(\sum_{j=1}^{L} a_{n,j} b_{nL+j} \right)$$

converges absolutely.

Proof: By replacing the b_j with $b_j - c$ we may assume without loss of generality that $b_j \searrow 0$. Also, we may assume $M = 1$. It suffices to prove convergence only, since we are free to change the signs of all the $a_{n,j}$ in a given segment without affecting the hypotheses.

Introduce the signed binary expansions

$$a_{n,j} = \sum_{k=1}^{\infty} \epsilon_{n,j,k} 2^{-k}, \ \epsilon_{n,j,k} \in \{\pm 1\}.$$

Define

$$\mu_{n,k} = \sum_{j=1}^{L} \epsilon_{n,j,k},$$

and

$$\lambda_{nL+j}^{(k)} = (\epsilon_{n,j,k} - \mu_{n,k})2^{-k} b_{nL+j}.$$

The numbers $(\epsilon_{n,j,k} - \mu_{n,k})$ are each integers in the range $\{-L-1, \dots L+1\}$, and they sum to zero on j. Thus, if each term of the series

$$\sum_{n=0}^{\infty} \sum_{j=1}^{L} \lambda_{nL+j}^{(k)}$$

is replaced by $|\epsilon_{n,j,k} - \mu_{n,k}|$ equal terms, the resulting series is segmentally alternating with segment length bounded by $L(L+1)$. Moreover, it is easy to see that the hypotheses (2.1) and (2.2) of Theorem 2.2 hold for the resulting series: we may use $L_k = b_1 2^{-k}$ in (2.1) and $R_k = L(L+1)$ in (2.2).

It follows then from Theorem 2.2 that the series

$$\sum_{n=1}^{\infty} \sum_{j=1}^{L} \sum_{k=1}^{\infty} (\epsilon_{n,j,k} - \mu_{n,k})2^{-k} b_{nL+j} = \sum_{n=1}^{\infty} \sum_{j=1}^{L} \left(a_{n,j} - \sum_{k=1}^{\infty} \mu_{n,k} 2^{-k} \right) b_{nL+j}$$

converges. But then we are done, since

$$\sum_{k=1}^{\infty} \mu_{n,k} 2^{-k} = \sum_{k=1}^{\infty} \sum_{j=1}^{L} \epsilon_{n,j,k} 2^{-k} = \sum_{j=1}^{L} a_{n,j} = 0.$$

References

[1] T. Apostol, *Mathematical Analysis*, Addison Wesley, Reading, MA, 1974.

[2] V.I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics,* Benjamin, New York, 1968.

[3] R. Goldberg, *Methods of Real Analysis*, 2nd Edition, Wiley, New York, 1976

[4] H. Katsuura, Extending the Alternating Series Test, *College. Math. J.*,**43**(2012), 325-330.

[5] M. Lothaire, *Combinatorics on Words*, Cambridge Mathematical Library, Cambridge, UK, 1997.

[6] Michael Schramm, John Trautman, and Daniel Waterman, Segmentally Alternating Series, *Amer. Math. Monthly***121**(2014),717-721.